Strategic Decentralized Matching: The Effects of Information Frictions*

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Abstract

We study strategic interactions in decentralized matching markets, where firms make directed offers to workers and agents' preferences are aligned. We show that implementing stable outcomes through decentralized interactions is possible, albeit under stringent conditions. Stable implementation in decentralized markets requires either complete information of preferences; no time frictions; or sufficient richness of plausible market realizations. Unique implementation occurs under even harsher restrictions on market interactions.

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1 Introduction

1.1 Overview

We study decentralized strategic interactions in matching markets, allowing for both information and time frictions. While the theoretical literature on two-sided matching markets has focused predominantly on analyzing outcomes generated through centralized clearinghouses, many markets are not fully centralized and have participants interacting in a less structured fashion: college admissions in the U.S., numerous labor markets such as the market for law clerks or junior economists, and others. Furthermore, almost all centralized markets are preceded or followed by decentralized matching opportunities for participants.[1] Understanding outcomes generated by decentralized markets is therefore important for the design of decentralized and centralized institutions alike. This paper offers a step in that direction.

Empirically, persistent centralized clearinghouses are associated with stable outcomes, see Roth (1991).[2] A common folk argument contends that if an unstable outcome is implemented, decentralized interactions, either preceding or following the operation of the clearinghouse, would ultimately yield stability; see Roth and Sotomayor (1992). Such decentralized interactions may come at an efficiency cost when frictions are present. For example, in labor markets, employers and employees already matched, who potentially have preferable matches elsewhere, would need to seek those out and take costly steps to undo their current arrangements. In this paper, we theoretically inspect this folk wisdom. We provide conditions under which (non-cooperative) decentralized interactions yield stability. In particular, we illustrate how both information and time frictions not only affect outcomes' efficiency, but also introduce critical obstacles to stability.

We suggest a simple model of a decentralized *market game* in which firms make directed offers to workers. Our market game is identified by three components: the preference distribution of agents (workers and firms), the information agents have about their own and others' realized preferences, and the extent of time frictions in the economy.

For simplicity, we focus on markets in which firms can employ up to one worker, who can work for at most one firm. We assume all agents are acceptable: any match is preferred to no match at all. We consider environments in which there is a unique stable matching. We do so for two

[1] See Roth (1984) and Roth (2008). For some evidence on outcome the large differences between centralized and decentralized markets, see Fréchette, Roth, and Ünver (2007) and Niederle and Roth (2003).

[2] A *stable matching* is a pairing of workers and firms in which no firm (worker) who is matched to a worker (firm), prefers to be alone, and no firm and worker pair prefer to jointly deviate by matching to one another.

reasons. First, uniqueness allows us to sidestep coordination problems, thereby eliminating one straightforward hurdle to stability. Second, previous work suggests that some markets exhibit only a few possible stable matchings, see Roth and Peranson (1999). In fact, recent theoretical work suggests that, under some conditions, large markets have a small number of stable matchings (see Immorlica and Mahdian 2005, Kojima and Pathak 2009, and Ashlagi, Kanoria, and Leshno 2017). As many of the leading applications of matching theory—from labor markets to school allocations—involve hundreds of thousands of participants, these results are in line with our uniqueness assumption.

Specifically, we concentrate our analysis on markets with *aligned preferences*, a restriction guaranteeing uniqueness of stable matchings. Aligned preferences require that the preferences of firms and workers can be represented by a joint "ordinal potential," analogous to ordinal potentials of normal-form games (Monderer and Shapley, 1996). An ordinal potential specifies a value for each firm and worker *pair* and captures the full preference profile of market participants. Preference alignment generalizes many restrictions the literature has offered to establish uniqueness of stable matchings. In fact, many prominent cases studied in the literature entail aligned preferences. For instance, alignment is ensured whenever firms and workers split revenue in fixed proportions, or when all participants on one market side rank participants on the other side in the same way.

In our decentralized market game, each firm can make up to one offer each period if she does not already have an offer held by some worker.[3] Workers can accept, reject, or hold an offer, where holding an offer ensures its availability in the future. A worker can accept at most one offer, but can hold any number of offers. To study the impacts of time frictions, we assume firms and workers receive discounted match utilities when they are matched.

We study two settings that differ in the initial information—complete or incomplete—that agents have about their own and others' preferences. Under *complete information*, all market participants are fully informed of the realized match utilities. As mentioned above, the existing literature has mostly focused on complete information environments (for a few exceptions, see the literature review below). However, allowing for incomplete information is particularly important from an applied perspective. Many matching markets are large and entail limited information sharing between participants. For instance, the job market for economists includes several thousands of applicants and new positions every year, the National Resident Matching Program (NRMP) involves around 70,000 participants annually, etc. Participants cannot possibly communicate their complete

[3] Throughout, we refer to firms as female and workers as male.

preference rankings to all others. Our *incomplete information* environment aims to capture such scenarios: each agent is fully informed only of her own match utilities and holds (correct) prior belief distributions pertaining to others' preferences.

We analyze equilibria in weakly undominated strategies. Under complete information, all agents can compute the stable matching. The stable outcome is, in fact, an equilibrium outcome of the decentralized market game. Still, there may be equilibria that yield unstable matchings, highlighting a first departure from the case in which agents participate in a centralized market, where all equilibria in weakly undominated strategies yield the stable matching. At the heart of this equilibrium multiplicity is the observation that in dynamic decentralized markets, agents can condition their actions on past activity. Nonetheless, a simple refinement—namely, iterated elimination of weakly dominated strategies—restores uniqueness of the stable matching as an equilibrium outcome.

With incomplete information, in a frictionless economy, stable matchings can still be implemented as an equilibrium outcome. Firms and workers can replicate, in essence, the firm-proposing Gale-Shapley deferred acceptance algorithm (Gale and Shapley, 1962) as part of an equilibrium profile. That is, firms make offers to workers in order of their preferences, and workers accept offers when they are made by their most preferred firms. Aligned preferences ensure there is always a firm and a worker that are each other's favorite partner. This guarantees that at least one firm and worker are matched in every period. In particular, such a strategy profile leads to a stable market outcome in finite time.

Nevertheless, even with minimal time frictions, agents may have incentives to deviate from these strategies. If all firms and workers emulate the firm-proposing deferred acceptance algorithm, the timing of events—offers, responses to offers, and exits—feeds into the updating process. There are then two types of manipulations: ones intended to speed up the matching process, and ones intended to affect participants' inferences about their expected partners. Consequently, in some economies, *no equilibrium* yields the stable outcome in all the supported markets.

To glean intuition for manipulations driven by a desire to speed up matches, consider an economy with two workers, Adrian and Bailey, and two firms, A and B. Suppose that (i) both workers prefer firm A to firm B and (ii) both firms prefer Adrian to Bailey. Under deferred acceptance strategies, both firms approach Adrian, their favorite, in the first period. Because Adrian prefers firm A, Adrian accepts firm A's offer, and they match. Bailey then receives an offer from firm B only in the second period. Foreseeing, Adrien and firm A's pairing, Bailey and firm B

realize that they are destined to match. If firm B proceeds out of order and makes an offer to Bailey in the first period, Bailey should accept that offer immediately. In this case, firm B successfully *speeds up* her match.

To understand what underlies manipulations designed to alter beliefs, consider a different economy. Suppose that, using deferred acceptance strategies, some agent, Cooper, matches with some firm C only when firm C makes an offer to Cooper in the first period. If there are markets in the economy in which Cooper is preferable to firm C's stable match partner, firm C has incentives to make an immediate offer to Cooper. An early offer to Cooper allows firm C to manipulate Cooper's beliefs about the realized market and secure a superior match.

We illustrate the hurdles to achieving stability in economies composed of aligned markets and patient agents. Without alignment, the possibility of coordination failures and the potential for rejection cycles pose further challenges to stability. With impatient agents, speeding up incentives can cause instability: highly impatient agents are willing to accept any partner who is preferable to staying unmatched. Either misalignment or impatience make stability all the more elusive.

In order to implement stable outcomes through equilibrium play, the learning that occurs from the timing of events per se has to be limited. Certainly, when information is complete, there are no learning opportunities and, indeed, the stable outcome can be implemented in equilibrium. We show that when the economy is sufficiently rich, when *any* aligned preference profile is possible with some positive probability, learning is highly restricted. In rich economies, there are many ways to rationalize various timing of events through the supported preference profiles. Consequently, timing plays a more limited role in learning. Equilibrium implementation of stable outcomes is then possible when agents are sufficiently patient.[4]

Since many applications pertaining to our model involve numerous participants, one may worry that the equilibria we identify demand a long time for markets to stabilize. Using simulations of randomly generated preferences, we show that the duration required for markets of reasonable size to culminate in a stable outcome is limited. Furthermore, these equilibrium dynamics take substantially shorter times than naïve dynamics yielding stability, á la Roth and Vande Vate (1990).

Taken together, our results illustrate that implementing stable outcomes through decentralized interactions is possible, albeit under stringent conditions. In the two extremes, when information is either complete or, in rich economies, profoundly incomplete, the precise sequencing of market interactions affords no learning opportunities. Stable matchings can then be implemented in

[4] In the Appendix, we provide more general sufficient conditions for implementation of stable matchings.

equilibrium, and our framework offers a non-cooperative foundation for the cooperative notion of stability. However, without richness, incomplete information and time frictions are detrimental to stability. In such environments, centralized interventions may be beneficial if stability is a goal.

1.2 Related Literature

The recent and growing literature on dynamic matching considers dynamic interactions when the set of market participants evolves, see e.g., Akbarpour, Li, and Gharan (2020), Baccara, Lee, and Yariv (2020), and the survey by Baccara and Yariv (2022). In that literature, market participants arrive over time—as is the case for organ patients and donors, children relinquished for adoption and parents seeking to adopt, etc. The focus is often on optimal clearinghouses for such settings, rather than outcomes of decentralized interactions that we consider. Several theoretical investigations inspect market outcomes as consequences of a dynamic interaction between a fixed set of market participants. Roth and Vande Vate (1990) and Ackermann et al. (2011) offer non-strategic dynamics of blocking-pair formation that yield stability, which we return to in Section 4.5. Haeringer and Wooders (2011) and Pais (2008) consider the case of complete information, and restrict firms' strategies by preventing offers to workers who had rejected them previously. Haeringer and Wooders (2011) study a game similar to ours in which firms can only make exploding offers that have to be accepted or rejected right away and show that in many environments equilibrium outcomes are stable. Pais (2008) investigates a setting with one firm chosen randomly each period to make an offer. She characterizes the set of (ordinal) subgame perfect equilibria and shows that outcome multiplicity may arise even when the underlying market has a unique stable matching. Suh and Wen (2008) consider a particular sequential protocol of offers in which each agent makes only one decision—accept a previous offer, stay single, or make an offer to someone who follows them. With complete information, they show conditions under which subgame perfect equilibrium outcomes are stable. We add to this line of work by considering markets in which both firms and workers interact strategically. We also consider both complete- and incomplete-information settings.[5]

Recent empirical work indicates the relevance of incomplete information. Grenet, He, and

[5] Dai and Jordan (2021) take an algorithmic approach for approximating payoff-maximizing strategies in decentralized markets. Arnosti, Johari, and Kanoria (2014) study decentralized matching markets in which agents arrive and depart asynchronously. They study the effects of observing who is available. The role of commitment in complete-information dynamic games is highlighted in Diamantoudi, Miyagawa, and Xue (2015) and Blum, Roth, and Rothblum (1997). There is also some work analyzing endogenous salaries in decentralized markets with complete information and limited dynamics, see Konishi and Sapozhnikov (2008).

Kübler (2021) consider a decentralized phase in Germany's university admission system and find evidence for students' costly learning about universities. Echenique et al. (2022a) show the importance of the decentralized interview process preceding the US centralized medical match. Using experiments, Pais, Pintér, and Veszteg (2020), Echenique, Robinson-Cortes, and Yariv (2022b), and Agranov et al. (2022) consider selection of and convergence to stable matchings in lab decentralized markets; see also references therein. In line with our results, complete-information markets often culminate in stable matchings, but that is not the case with incomplete information.

Most of the theoretical matching literature effectively assumes complete information of preferences. Liu et al. (2014) and Liu (2020) suggest a cooperative notion of stability with incomplete information allowing for transfers, while Bikhchandani (2017) offers a notion absent transfers. Roth (1989) and Fernandez, Rudov, and Yariv (2022) illustrate the impacts of incomplete information in a centralized clearinghouse; see also references there. We add to this literature by identifying environments in which, even with incomplete information, non-cooperative stabilization dynamics yield complete-information stable outcomes in equilibrium.

The search and matching literature—Burdett and Coles (1997), Eeckhout (1999), Shimer and Smith (2000), and the survey by Chade, Eeckhout, and Smith (2017)—studies dynamic interactions in an often stationary market in which agents are commonly evaluated. As in our setting, time frictions present a stability hurdle: they impose a search cost that leads agents to "compromise". However, in those models, as time frictions vanish, stability is approximated. We consider a richer class of preferences and show that, with incomplete information, even vanishingly small time frictions can present a severe impediment to stability unless learning opportunities are limited.

Gutin, Neary, and Yeo (2021) characterize preference restrictions guaranteeing a unique stable matching. Preference alignment is somewhat more restrictive, but reminiscent of many identified uniqueness conditions (Clark 2006 and Eeckhout 2000). Some of its special cases often appear in empirical work. For example, a similar assumption is made in Agarwal (2015) when analyzing the medical match and in Dur et al. (2018) when examining a school-choice setting in Taiwan.

2 The Model

2.1 The Economy

A *market* is a triplet $\mathcal{M} = (\mathcal{F}, \mathcal{W}, U)$, where $\mathcal{F} = \{1, \ldots, F\}$ and $\mathcal{W} = \{1, \ldots, W\}$ are disjoint sets of firms and workers, respectively, and $U = \left\{\left\{u_{ij}^{f}\right\}, \left\{u_{ij}^{w}\right\}\right\}$ are agents' match utilities. Each firm $i \in \mathcal{F}$ receives match utility u^f_{ij} from matching with worker $j \in \mathcal{W} \cup \varnothing$, where matching to $\varnothing$ is interpreted as remaining unmatched. Similarly, each worker $j \in \mathcal{W}$ receives match utility u^w_{ij} from matching with firm $i \in \mathcal{F} \cup \varnothing$. We denote by $U^f = \left(u^f_{ij}\right)_{i \in \mathcal{F}, j \in \mathcal{W}}$ and $U^w = \left(u^w_{ij}\right)_{i \in \mathcal{F}, j \in \mathcal{W}}$ the matrices corresponding to utilities from firm-worker pairs for both sides of the market.

For simplicity, we assume that all match utilities are strictly positive, that firms and workers have strict preferences, and that all agents prefer to be matched over remaining unmatched.[6] That is, for any firm i and workers $j, j' \in \mathcal{W} \cup \varnothing$,

$$u^f_{ij} \neq u^f_{ij'} > u^f_{i\varnothing} > 0.$$

Similarly, for any worker j and firms $i, i' \in \mathcal{F} \cup \varnothing$,

$$u^w_{ij} \neq u^w_{i'j} > u^w_{\varnothing j} > 0.$$

In particular, all outside options generate strictly positive utilities.[7]

For fixed sets $\mathcal{F}$ and $\mathcal{W}$ of firms and workers, an *economy* is a finite collection of markets $\{(\mathcal{F}, \mathcal{W}, U)\}_{U \in \mathcal{U}}$ together with a distribution G over possible match utilities $U \in \mathcal{U}$. We assume G has finite support $\mathcal{U}$: each market in the economy has positive probability and cannot be ignored in agents' considerations.

A *matching* is a function $\mu : \mathcal{F} \cup \mathcal{W} \to \mathcal{F} \cup \mathcal{W} \cup \varnothing$ such that for all $i \in \mathcal{F}$, $\mu(i) \in \mathcal{W} \cup \varnothing$, and for all $j \in \mathcal{W}$, $\mu(j) \in \mathcal{F} \cup \varnothing$. Furthermore, if $(i, j) \in \mathcal{F} \times \mathcal{W}$, then $\mu(i) = j$ if and only if $\mu(j) = i$. If $\mu(k) \neq \varnothing$ for $k \in \mathcal{F} \cup \mathcal{W}$, we say that k is matched under μ. A *blocking pair* for a matching μ is a pair $(i, j) \in \mathcal{F} \times \mathcal{W}$ such that $u^f_{ij} > u^f_{i\mu(i)}$ and $u^w_{ij} > u^w_{\mu(j)j}$. A matching is *stable* if there exist no blocking pairs. Since all agents prefer to be matched, individuals never block a matching.

As noted in our literature review, in a static setting, one could consider an adjusted notion of stability that allows for incomplete information; see, e.g., Liu et al. (2014) and Bikhchandani (2017). Our goal here is to understand when learning in matching markets can generate the complete-information stable outcomes. When it does, outcomes are expected to be robust to any further information or interactions after the dust has settled in the market. Throughout the paper, we slightly abuse terminology and refer to the complete-information stable matchings in a market as stable matchings.

[6] These assumptions simplify our presentation. Relaxing either would not alter our qualitative conclusions.

[7] With an outside option that generates a utility of 0, agents expecting to exit do not care about the timing of their exit, which would generate equilibrium multiplicity mechanically.

Gale and Shapley (1962) showed that any market has a stable matching, and provided an algorithm that identifies one. In the *firm-proposing deferred acceptance (DA) algorithm*, in step 1, each firm makes an offer to its most preferred worker. Workers collect offers, hold the offer from their most preferred firm, and reject all other offers. In a general step k, each firm whose offer was rejected in the last step makes an offer to her most preferred worker who has not rejected her yet. Workers once more collect offers including, possibly, an offer held from a previous step, keep their most preferred offer, and reject all other offers. The algorithm ends when there are no more offers that are rejected; that is, when each firm either has her offer held by a worker, or has been rejected by all workers. Once the algorithm ends, held offers are converted to matches.

While the firm-proposing DA algorithm generates one stable matching, the one preferred by all firms, there may be others. Certainly, in markets with multiple stable matchings, learning to coordinate on a particular stable matching for any market realization would entail many difficulties. For one, agents need to learn which stable matching is being played, in addition to various relevant features of the market. Furthermore, incentive compatibility concerns akin to those emerging in centralized settings with multiple stable matchings may surface in decentralized markets as well (see Roth and Sotomayor 1992, and Section 2.4 below). In order to circumvent these hurdles and isolate the impacts of incomplete information and frictions on learning in matching markets, we assume that any market $\mathcal{M} = (\mathcal{F}, \mathcal{W}, U)$ in the support of G has a unique stable matching denoted by μ_M. This assumption is in line with prior work. For instance, Roth and Peranson (1999)'s analysis of the medical residents' match data finds small cores. As already mentioned, recent theoretical work (see Immorlica and Mahdian 2005, Kojima and Pathak 2009, and Ashlagi, Kanoria, and Leshno 2017) supports the idea that, in applications that involve a large participant volume, there is a small number of stable matchings.

2.2 Aligned Preferences

We focus on a class of preferences we term *aligned preferences* that exhibit a unique stable matching and several other desirable features.

Definition (Aligned Preferences)**.** Firms and workers have *aligned preferences* if there exists an *ordinal potential* $\Phi = \left(\Phi_{ij}\right)_{i \in \mathcal{F}, j \in \mathcal{W}}$, $\Phi_{ij} \in \mathbb{R}$ such that for any workers $j, j' \in \mathcal{W}$ and firms $i, i' \in \mathcal{F}$:

$$u^w_{ij} > u^w_{i'j} \iff \Phi_{ij} > \Phi_{i'j} \quad \text{and} \quad u^f_{ij} > u^f_{ij'} \iff \Phi_{ij} > \Phi_{ij'}.$$

Intuitively, preference alignment imposes a link between firms' and workers' (ordinal) preferences through the ordinal potential. The preference ranking of both sides of the market can be captured using one common matrix Φ. Let $\hat{U}^f = \hat{U}^w = \left(\Phi_{ij}\right)_{i\in\mathcal{F},j\in\mathcal{W}}$. Then, $\hat{U}^f$ and $\hat{U}^w$ capture the same ordinal preferences over partners as U^f, U^w.[8] Many applied papers implicitly assume that preferences are aligned (e.g., Sørensen 2007, Agarwal 2015, and Dur et al. 2018).

Several special cases of preference alignment are prominent in the literature. For instance, suppose firms and workers have a joint production output they share in fixed proportions when matched. That is, for $\alpha > 0$ and all $(i,j) \in \mathcal{F}\times\mathcal{W}, u^w_{ij} = \alpha u^f_{ij} > 0$. Here, $\Phi = \left(\Phi_{ij}\right)_{i\in\mathcal{F},j\in\mathcal{W}}$ defined as $\Phi_{ij} = u^f_{ij}$ for all i and j serves as an ordinal potential. Alternatively, suppose participants on one side of the market rank participants on the other side of the market identically—firms may all value potential employees using their college GPA, or students applying to colleges may all use the *U.S. News and World Report* rankings to form their preferences. Such preferences are also aligned.[9]

As we now describe, preference alignment implies uniqueness of the stable matching, in addition to several properties useful for our analysis.

Define a market $(\tilde{\mathcal{F}}, \tilde{\mathcal{W}}, \tilde{U})$ as a *submarket* of $(\mathcal{F}, \mathcal{W}, U)$ if $\tilde{\mathcal{F}} \subseteq \mathcal{F}$, $\tilde{\mathcal{W}} \subseteq \mathcal{W}$, and $\forall i, j \in (\tilde{\mathcal{F}} \cup \varnothing) \times (\tilde{\mathcal{W}} \cup \varnothing) \setminus \{\varnothing, \varnothing\}$, $\tilde{u}^w_{ij} = u^w_{ij}$ and $\tilde{u}^f_{ij} = u^f_{ij}$. A direct implication of preference alignment is that, for any submarket $(\tilde{\mathcal{F}}, \tilde{\mathcal{W}}, \tilde{U})$, there is a firm i and a worker j that form a *top-top* match, i.e. worker j is firm i's most preferred worker within $\tilde{\mathcal{W}}$ and firm i is worker j's most preferred firm within $\tilde{\mathcal{F}}$. Indeed, when preferences are aligned, any submarket has an ordinal potential. Suppose $\tilde{\Phi}$ is an ordinal potential of the submarket $(\tilde{\mathcal{F}}, \tilde{\mathcal{W}}, \tilde{U})$. Consider a pair $(i,j) \in \arg\max_{(i',j')\in\tilde{\mathcal{F}}\times\tilde{\mathcal{W}}} \tilde{\Phi}_{i'j'}$. It follows that (i,j) is a top-top match in $(\tilde{\mathcal{F}}, \tilde{\mathcal{W}}, \tilde{U})$. When every submarket has a top-top match, we say that preferences satisfy the **top-top match property**. Preference alignment implies the top-top match property.[10]

The top-top match property guarantees uniqueness of the stable matching. Indeed, find the firm-worker pairs that constitute top-top matches, pairs (i,j) at which the ordinal potential $\Phi = \left(\Phi_{ij}\right)$ is maximized. A firm and worker in such a pair are each other's favorites in the market and must be matched in any stable matching. The remaining firms and workers form a submarket

[8] The notion of ordinal potential is analogous to that of a potential in two-player games in which agents' match utilities replace the payoff matrix $\left(U^w, U^f\right) = \left((u^w_{ij}, u^f_{ij})\right)_{i\in\mathcal{F},j\in\mathcal{W}}$, see Monderer and Shapley (1996).

[9] Suppose, without loss of generality, that firms share a single preference ranking over workers. Order the workers according to this ranking, with worker 1 being the least preferred worker and worker n being the most preferred worker. Normalize each worker's match utilities to $\tilde{u}^w_{ij}$, such that for all $i, i' \in \mathcal{F}, j \in \mathcal{W}$: $0 < \tilde{u}^w_{ij} < 1$, and $\tilde{u}^w_{ij} < \tilde{u}^w_{i'j} \Leftrightarrow u^w_{ij} < u^w_{i'j}$. Then $\Phi = \left(\Phi_{ij}\right)_{i\in\mathcal{F},j\in\mathcal{W}}$ with $\Phi_{ij} = j + \tilde{u}^w_{ij}$ for all i, j is an ordinal potential.

[10] The top-top match property is referred to as α-reducibility in Clark (2006).

with aligned preferences. Continuing recursively achieves the unique stable matching.

Suppose μ_M is the unique stable matching of market $M = (\mathcal{F}, \mathcal{W}, U)$ with aligned preferences. Then, the **stable blocking pair property** holds. That is, for any unstable matching μ, there exists a *stable blocking pair*: a blocking pair (i, j) for which $\mu_M(i) = j$. Indeed, suppose $\mu \neq \mu_M$. Consider the recursive process described above to illustrate uniqueness. At some stage, a discrepancy must arise between μ_M and μ. At that stage, a match that occurs under μ_M does not get formed. The corresponding worker and firm form a stable blocking pair.

Finally, preference alignment implies that when firms make offers in the order of their preferences, a rejected offer of a worker cannot trigger a chain of offers and rejections that results in an offer from a more desirable firm. That is, if a worker j rejects an offer from firm i, the resulting chain of offers can only result in offers to worker j that he prefers less than the original offer from firm i. Formally, there is no sequence $i_1, ..., i_n \in \mathcal{F}$ and $j_1, ..., j_n \in \mathcal{W}$ such that, by rejecting firm i_2, worker j_1 can trigger a chain of offers yielding an offer from a preferred firm i_1 :

$$u^w_{i_1 j_1} > u^w_{i_2 j_1}, \;\; u^f_{i_2 j_1} > u^f_{i_2 j_2}, \;\; u^w_{i_2 j_2} > u^w_{i_3 j_2}, \;\; u^f_{i_3 j_2} > u^f_{i_3 j_3}, ..., \;\; u^w_{i_n j_n} > u^w_{i_1 j_n}, \;\; u^f_{i_1 j_n} > u^f_{i_1 j_1}$$

Such a chain would be equivalent to having a cycle in the payoff matrix $\left(U^w, U^f\right)$. We thus say that preferences satisfy the **no-cycle property**. The characterization of potential games by Voorneveld and Norde (1997) ensures that preference alignment is *equivalent* to the no-cycle property.

Proposition 1 summarizes the properties of markets with preference alignment.

Proposition 1 (Alignment – Properties)**.**

1. *If preferences are aligned, the stable matching is unique. Furthermore, the top-top match and stable blocking pair properties hold.*

2. *Preferences are aligned if and only if the no-cycle property holds.*

2.3 A Decentralized Market

For a given economy $\{(\mathcal{F}, \mathcal{W}, U)\}_{U \in \mathcal{U}}$ and a distribution G over utility realizations (or markets), we analyze the following *market game*. The economy, together with the distribution G, are common knowledge to all agents. At the outset, the market is realized according to the distribution G. Each agent is privately informed only of their own realized match utilities. When the support of G is a singleton, there is *complete information*: firms' and workers' match utilities are common knowledge. This is the case that most of the literature has tackled. With complete information, both firms and workers can deduce the stable outcome. We also analyze settings in which the support of G is non-trivial. In this *incomplete information* case, participants may not be able to infer the underlying market, nor its stable matching, from their private information.

In our market game, firms make offers over time, indexed by $t \in \{1, 2, ...\}$, and workers react to those offers. Specifically, each period has three stages. In the first stage, eligible firms simultaneously decide whether and to whom to make an offer and whether to exit the market. In the second stage of any period, workers observe which firms exited, and the offers they themselves received. Each worker j who has received an offer from firm i can accept, reject, or hold the offer. If the offer is accepted, worker j is matched to firm i. A worker can accept at most one offer, but hold any number of offers. Workers can also exit the market. In the third stage, firms observe rejections and deferrals of their own offers. Finally, all participants are informed of the agents who exited the market and the participants who were matched.[11]

Eligible firms are firms that have not yet hired a worker and have no offer held by a worker. In each period t, eligible firms can make up to one offer to any worker that has not yet been matched. The game ends once all participants have exited the market, matched or unmatched.

We consider market games with and without time frictions, which take the form of discounting. If a firm i is matched to worker j at time t, firm i receives $\delta^t u_{ij}^f$ and worker j receives $\delta^t u_{ij}^w$, where $\delta \in [0, 1]$ is the market discount factor. As long as agents are unmatched, they receive 0 in each period. One interpretation is that once a worker and a firm are matched, they receive their match utility or, equivalently, they receive a constant, perpetual stream of payoffs, the present value of which is their match utility. One can also interpret the discount factor δ as the probability of market collapse, or the probability that each firm loses its position and receives a payoff of 0 (and, analogously, the probability that each worker leaves the market and receives 0 as well).[12]

We assume all offer benefits are captured by match utilities. In particular, firms do not offer targeted transfers or wages to workers. Hall and Krueger (2012) suggest that a substantial fraction of jobs entails posted wages, plausibly resistant to general equilibrium forces. Our analysis pertains to such settings. With incomplete information, targeted transfers may introduce additional obstacles to stability: market participants then need to infer not only their correct match partner, but also

[11]Participants do not observe the entire stream of offers and responses made by others, only their matches and exits. This assumption fits many applications and makes the belief-updating process tractable.

[12]An alternative source of frictions would be costly offers. An analogous model in which each offer costs a constant amount would yield qualitatively similar results to those presented here.

the appropriate wage.[13]

The equilibrium notion we use is that of Bayesian Nash equilibrium, where types correspond to agents' private information. When the support of G is a singleton and information is complete, our analysis pertains to the Nash equilibria of the corresponding game.

We focus on equilibria of decentralized market games in which all agents use weakly undominated strategies. This imposes several restrictions on equilibrium play:

1. A worker who accepts an offer always accepts his best available offer. In particular, a worker cannot exit and simultaneously reject an offer since offers always generate higher payoffs.

2. When $\delta < 1$, a worker who receives an offer from his most preferred unmatched firm accepts it immediately. Similarly, if only agents on one side of the market are unmatched, they exit immediately since the payoffs from this outside option are strictly positive.

2.4 Centralized Matching Benchmark

Our decentralized market game shares features with the firm-proposing DA algorithm: firms make at most one offer at any time and operate in stages; workers can hold on to offers until they are content with one that they accept. There are important differences since market participants' strategies are unrestricted. Firms need not go in order of their preference ranking and can make repeat offers; workers can hold multiple offers, and accept or reject any of them. Furthermore, dynamic interactions allow for non-trivial conditioning on and learning from observed histories. Nonetheless, the centralized clearinghouse utilizing the firm-proposing DA is a natural benchmark.

To ease analogies between centralized and decentralized markets, we assume that agents in a centralized market report match utilities that are then converted into ordinal preferences. That is, each agent submits a vector of positive match utilities.[14]

Formally, for each type of agent $\alpha \in \{f, w\}$, and each agent l, let $P(u_l^\alpha)$ be the strict ordinal preferences associated with l's reported match utilities, in which ties are broken by favoring lower-indexed match partners and by favoring being matched.[15] Specifically, consider firm i, then

[13] Agranov et al. (2022) report experimental results from decentralized markets with complete and incomplete information. Identifying stable transfers is difficult for participants, particularly with incomplete information.

[14] The restriction to positive numbers is made only for presentation simplicity. It suffices that the set of available reports contains as many elements as the maximal number of different match utility vectors possible in the economy.

[15] Although agents are never indifferent in their realized match utilities, they may still report indifferences.

$u^f_{ij} > u^f_{ik}$ implies $jP(u^f_{i\cdot})k$, for $j,k \neq \emptyset$, $u^f_{ij} = u^f_{ik}$ and $j < k$ implies $jP(u^f_{i\cdot})k$, and for $j \neq \emptyset$, $u^f_{ij} \geq u^f_{i\emptyset}$ implies $jP(u^f_{i\cdot})i$. Any worker's preferences are constructed analogously.

We define a *deferred acceptance (DA) mechanism* as a mechanism in which all agents report their match utilities simultaneously, after receiving private information. The mechanism then computes the corresponding ordinal preferences as above and outputs the stable matching induced by the firm-proposing DA algorithm. The payoffs of firms and workers are the match utilities corresponding to the implemented matching.

It follows directly from incentive compatibility attributes of the DA algorithm that the DA mechanism allows for a Bayesian Nash equilibrium in weakly undominated strategies in which the resulting matching corresponds to the unique stable matching in each market of the economy (see Roth and Sotomayor, 1992). That is,

Lemma 1 (Centralized Matching)**.**

1. *For any complete-information economy, all Nash equilibria in weakly undominated strategies of the game associated with the DA mechanism yield the unique stable matching.*
2. *For any economy, there exists a Bayesian Nash equilibrium in weakly undominated strategies of the game associated with the DA mechanism that yields the unique stable matching in each market.*

Although implementing the stable matching is possible through a centralized clearinghouse whenever there is a unique stable matching (even absent alignment), the stable matching is not necessarily the unique equilibrium outcome with incomplete information (Fernandez et al. 2022). As we show, equilibrium multiplicity is even more more severe in decentralized markets, where further refinements are needed to guarantee uniqueness, even when information is complete.

3 Decentralized Complete Information Economies

We start by analyzing our decentralized market game in economies in which all participants are informed of the details of the market, when the support of G is a singleton. In this case, all agents can compute the stable matching. The stable matching can then be achieved in one period. Intuitively, consider the following strategy profile. Each firm makes an offer to her stable match partner under the unique stable matching μ_M and exits the market if unmatched under μ_M. Each worker accepts his best available offer in period 1, and exits upon receiving no offers. This profile constitutes an equilibrium in weakly undominated strategies that yields the matching μ_M.

Ruling out weakly dominated strategies is not sufficient to guarantee uniqueness, however. First, there may be multiple equilibria generating μ_M. Indeed, for sufficiently high discount factors, an alternative way of implementing μ_M through equilibrium involves emulating the DA algorithm. Since this profile may entail several periods of market activity, it can generate different equilibrium payoffs for non-trivial discount factors.

Furthermore, there may be outcomes generated by equilibria in weakly undominated strategies that are unstable, as the following example illustrates.

Example 1 (Multiplicity with Complete Information) Assume an economy with four firms $\{f_1, f_2, f_3, f_4\}$ and four workers $\{w_1, w_2, w_3, w_4\}$. A firm and worker who match receive the same match payoffs, defined by the following matrix:

$$U^f = U^w = \begin{array}{|c|c|c|c|} \hline \mathbf{1} & \underline{2} & 7 & 4 \\ \hline \underline{5} & \mathbf{6} & 3 & 8 \\ \hline 9 & 10 & \underline{\mathbf{11}} & 12 \\ \hline 13 & 14 & 15 & \underline{\mathbf{16}} \\ \hline \end{array}$$

where bold entries correspond to the unique stable matching: $\mu_M(f_i) = w_i$ for all i. For sufficiently high $\delta < 1$, we show there is an equilibrium in weakly undominated strategies that implements the matching μ: $\mu(f_1) = w_2, \mu(f_2) = w_1, \mu(f_3) = w_3$, and $\mu(f_4) = w_4$, corresponding to the underlined entries in the matrix.

Consider the following profile of strategies yielding the matching μ. In period 1, firms f_2 and f_4 make an offer to workers $w_1 = \mu(f_2)$ and $w_4 = \mu(f_4)$, respectively. w_1 and w_4 accept these offers immediately, while workers w_2 and w_3 do not accept any offer, unless from their most preferred unmatched firm. In period 2, firms f_1 and f_3 make an offer to $w_2 = \mu(f_1)$ and $w_3 = \mu(f_3)$, respectively, who accept their offers.

With detectable deviations, if workers w_1 and w_4 do not receive an offer from any firm in period 1, they exit; otherwise, all workers reject any offer they receive, unless it is from their most-preferred firm, and stay in the market until period 2. In period 2, if firms f_2 and f_4 have not matched with $\mu(f_2)$ and $\mu(f_4)$, respectively, f_3 makes an offer to w_2 instead of $w_3 = \mu(f_3)$ if possible. If w_2 has already exited the market, f_3 makes an offer to its most preferred unmatched worker, and exits if all workers have left the market. Furthermore, in period 2, any worker who does not receive offers exits immediately, and otherwise accepts his best offer. Any firm rejected in period 1 exits the market at the start of period 2.

This profile constitutes an equilibrium in weakly undominated strategies that implements an unstable matching.[16] ∥

The crux of the multiplicity in the example above is the limited restraint that elimination of weakly dominated strategies provides on off-equilibrium behavior. Specifically, f_3 can "punish" f_2 for not making an offer to $w_1 = \mu(f_2)$ in period 1. Imposing subgame perfection would have little impact in our decentralized market game since individuals do not observe others' offers and reactions: the set of proper subgames is limited after period 1. Nonetheless, iterated elimination of weakly dominated strategies rules out strategies such as those driving Example 1 and, in fact, guarantees that the stable matching is the unique equilibrium outcome.

Proposition 2 (Complete Information)**.** *For any complete-information decentralized market game, there exists a Nash equilibrium in weakly undominated strategies that yields the stable matching. Furthermore, the stable matching is the unique Nash equilibrium outcome surviving iterated elimination of weakly dominated strategies.*

When using strategies that survive iterated elimination of weakly dominated strategies, firms and workers that form top-top matches must be matched in period 1. Consider the top-top matches in the remaining submarket. Since the corresponding workers realize that top-top matches in the original market are formed in period 1, iterated elimination of weakly dominated strategies ensures that they accept their top-top matches in the remaining submarket. Therefore, the corresponding firms make those offers and are matched in period 1. Continuing recursively, iterated elimination of weakly dominated strategies guarantees that the unique stable matching of the market is implemented in one period.[17]

The above construction hinges on all agents being informed of the realized market, and hence able to compute the stable matching. Proposition 2 shows that a robust non-cooperative outcome coincides with the unique stable matching in that case. In what follows, we show that this conclusion may change dramatically when information is incomplete and time frictions are present.

[16] When firms and workers can use weakly dominated strategies, there are even more equilibrium profiles and outcomes. For instance, it is an equilibrium for all agents to exit the market in period 1, resulting in no individual matches. Weak dominance rules this out, as it does not allow a worker to exit the market when he has an acceptable offer in hand.

[17] Preference alignment is important for Proposition 2. Without it, equilibria surviving iterated elimination of weakly dominated strategies may generate multiple outcomes, even with complete information.

4 Decentralized Incomplete Information Economies

In economies with incomplete information, for participants to reach a stable outcome, sufficient information has to be transmitted to ensure that (i) firms make offers to workers who are their stable match partners in the realized market, and (ii) workers only accept offers from firms that are their stable match partners in that market.

There are three channels through which information flows in the decentralized market we study. First, information is publicly transmitted when agents exit the market or form a match. Second, information is privately transmitted when workers receive offers from firms and workers respond to those offers (unless offers are accepted, in which case that information becomes public). The third component of information is time, which all market participants track.

4.1 The No-Discounting Case

Suppose there is no discounting: $\delta = 1$. One way by which information can be transmitted in the market is by agents mimicking the DA algorithm. That is, firms make offers to workers according to their match utilities, and exit the market only when all workers have rejected them. Workers hold on to their best available offer, and accept an offer from any firm that generates the highest match utility among the set of available firms. These prescriptions are followed by all agents after any detectable deviations as well, and hold independently of the initial distribution G. We call these **DA strategies**.

Proposition 3 (No Discounting)**.** *Suppose $\delta = 1$. The DA strategy profile constitutes a Bayesian Nash equilibrium in weakly undominated strategies and yields the stable matching. Furthermore, the stable matching in each of the economy's markets is the unique Bayesian Nash equilibrium outcome surviving iterated elimination of weakly dominated strategies.*

Intuitively, when agents use DA strategies, workers ultimately hold offers from their stable match partners. The top-top match property implies that, in every period, either a match is formed, or only agents on one side remain unmatched. Thus, the game terminates in finite time. When $\delta = 1$, the timing of matches is inconsequential to market participants and unilateral deviations cannot generate a preferable match. Certainly, the market could stall, with no agent acting, for long but finite periods. Furthermore, as Example 1 already illustrated, even uniqueness of the resulting matchings is not guaranteed without further refinements. Nonetheless, as with complete information, alignment guarantees the matching outcomes are unique when selecting Bayesian Nash equilibria surviving iterated elimination of weakly dominated strategies.

4.2 Instability in Economies with Discounting

When $\delta < 1$, DA strategies are in general no longer incentive compatible. As an example, consider a complete information economy with two workers and two firms, for which $u^f_{ij} = u^w_{ij}$ for all i, j and match utilities are given as follows:

$$U^f = U^w = \begin{array}{|c|c|} \hline \mathbf{4} & 1 \\ \hline 3 & \mathbf{2} \\ \hline \end{array}$$

The unique stable matching is highlighted in bold. Firm 2 knows that worker 2 is her unique stable match partner and, furthermore, that worker 2 would accept an offer from firm 2 immediately, as firm 2 is worker 2's first choice. Hence, it cannot be an equilibrium for firm 2 to first make an offer to worker 1 and delay her match. Firms may therefore be tempted not to make all offers in order of their preferences, but to instead concentrate on offers to plausible stable match partners. Similarly, workers may accept an offer from their highest plausible stable match partner, even if more preferred firms are still unmatched.

As it turns out, timing considerations are an inherent obstacle to stability in incomplete-information economies with frictions. The incentives to speed up matches and affect inferences may be so severe that *no* equilibrium generates the stable outcome in all markets in the economy's support, as the following example illustrates.

Example 2 (Instability through Belief Manipulation) Consider an economy with three firms $\{f_1, f_2, f_3\}$ and three workers $\{w_1, w_2, w_3\}$. Each of 4 markets occurs with positive probability. Match utilities in market $k = 1, \ldots, 4$ are given by $U^f_k = U^w_k = U_k$, with

$$U_1 = \begin{array}{|c|c|c|} \hline \mathbf{5} & 4 & 3 \\ \hline 7 & \mathbf{10} & 2 \\ \hline 6 & 8 & \mathbf{9} \\ \hline \end{array} \quad U_2 = \begin{array}{|c|c|c|} \hline 5 & 9 & \mathbf{4} \\ \hline 7 & \mathbf{10} & 3 \\ \hline \mathbf{6} & 8 & 2 \\ \hline \end{array} \quad U_3 = \begin{array}{|c|c|c|} \hline \mathbf{5} & 4 & 3 \\ \hline 2 & 6 & \mathbf{9} \\ \hline 1 & \mathbf{8} & 7 \\ \hline \end{array} \quad U_4 = \begin{array}{|c|c|c|} \hline \mathbf{5} & 4 & 3 \\ \hline 9 & 7 & \mathbf{10} \\ \hline 6 & \mathbf{8} & 2 \\ \hline \end{array}$$

Suppose that there exists δ^* such that, for any $\delta > \delta^*$, there exists an equilibrium in weakly undominated strategies that always implements the stable matching, highlighted in bold.

The specification of U_3 guarantees that, in any such equilibrium, f_1 with match utilities $(5, 4, 3)$ makes an offer to w_1 with probability 1.[18] Furthermore, the specification of U_4 guarantees that, for

[18] When U_3 prevails, w_1 accepts an offer from f_1 immediately. Discounting implies that making this offer is optimal.

sufficiently high δ, in any such equilibrium, firm f_3 with match utilities $(6,8,2)$ always makes an offer to worker w_2 in period 1.[19] From here on, we concentrate on the markets corresponding to U_1 and U_2.

When U_2 prevails, any equilibrium yielding the stable outcome in weakly undominated strategies must have f_1 making an offer to w_3 in period 1. Indeed, f_1 can foresees matching with w_3, who accepts an offer from her immediately; any delay would be suboptimal. Therefore, w_1 receives an offer from f_1 only when f_1 is his stable match partner: when U_2 prevails, f_3 makes an offer to w_2 in period 1 and to w_1 only in period 2. Thus, in any such equilibrium, w_1 accepts an offer from f_1 in period 1 if it is the only offer he receives. But then, f_1 can profitably deviate to making an offer to w_1 even when U_2 prevails, in contradiction.

In this example, w_1 cannot distinguish between U_1 and U_2. Hence, his set of plausible stable match partners at $t = 1$ is $\{f_1, f_3\}$ when either U_1 or U_2 govern match utilities. At the heart of the difficulty in achieving equilibrium stability is the fact that w_1 cannot be certain whether he will receive his best offer in period 1 or in period 2. He tries to infer that from offers received in period 1. The example illustrates the potential for belief manipulation when information regarding the set of potential stable match partners is transmitted by the mere timing of an offer (or its target's response). This form of information transmission needs to be restricted to allow for equilibria that yield the stable matching in every market. As it turns out, the addition of markets can help prevent this problem by limiting learning through the timing of events per se. For instance, suppose we add another market to the economy, described by match utilities $U_5^f = U_5^w = U_5$ as follows:

$U_5 =$

5	4	**3**
7	10	2
6	**11**	5

with an arbitrary, but strictly positive, probability of each of the 5 markets. In this augmented economy, for sufficiently high discount factors, f_2 first makes an offer to w_2 since she cannot tell whether U_1 or U_5 govern match utilities. Certainly, f_3 makes an offer to w_2 as well since she knows they form a top-top match. It follows that, when U_5 prevails, w_1 receives only one offer from f_1 at $t = 1$, which he should no longer accept. This breaks the incentive of f_1 to deviate when U_2 prevails, as in the original 4-market economy. ||

[19]Indeed, such an offer is immediately accepted when U_4 prevails. Furthermore, if f_3 makes an offer to w_1 with some probability $p > 0$ in period 1, then when U_4 prevails, since w_1 can infer his stable match partner is f_1 generating a match utility of 5, w_1 accepts the offer from f_3, yielding an unstable outcome. For sufficiently high δ, that cannot be optimal.

4.3 Rich Economies and Stable Equilibrium Outcomes

Example 2 suggests that obstacles to stability emerge when learning occurs via the timing of events. When all participants are informed of the realized market, timing bears no information as there is no learning to be had; stable outcomes can then be implemented in equilibrium. As Example 2 further illustrates, the addition of certain markets limits the scope for manipulation as well. We now show that, indeed, when there are many possible markets in the economy, learning through the timing of events is restrained and stability can be achieved in equilibrium.

We introduce *rich economies*, where the support of G contains markets representing all possible constellations of aligned preferences. In rich economies, an agent's utility type provides limited information regarding other agents' utility types. Furthermore, at the outset of the market game, all matchings are conceivably stable regardless of an agent's private information. As we show, in such economies, learning from the timing of events is restricted and stable outcomes can be implemented in equilibrium.

Formally, a **rich economy** is an economy in which the distribution G over markets has full support over all possible aligned preferences represented with payoffs from $\Pi \subseteq \mathbb{R}_{>0}$ with $|\Pi| \geq \max\{F, W\} + 1$.[20]

Richness ensures that every possible preference constellation associated with an aligned market has a positive probability. In particular, it drastically limits what agents can infer about others' preferences from their own realized match utilities. We stress that richness imposes no restrictions on the likelihood of any market in the economy, which can be arbitrarily small.

Proposition 4 (Stable Implementation in Decentralized Economies)**.** *Suppose the economy is rich. In the decentralized market game with sufficiently high δ, there exists a Bayesian Nash equilibrium in weakly undominated strategies that implements the unique stable matching in each supported market.*

We describe the intuition for the proof of Proposition 4 in Subsection 4.4 below. While we present Proposition 4 for rich economies, in the Appendix, we provide more general conditions on the support of the distribution G that allow for the implementation of stable outcomes. The conditions formalize the restrictions that, under a desirable class of strategy profiles that culminate in stable matchings, the timing of events does not reveal information in and of itself. For instance,

[20] The requirement that $|\Pi| \geq \max\{F, W\} + 1$ is needed since we assume agents are never indifferent. Indeed, suppose $\max\{F, W\} = F$. For each worker, we need to specify match utilities generated by all F firms, as well as the match utility from remaining unmatched, necessitating $F + 1$ values.

full support on all aligned markets in which preferences are "assortative" on one side—say, where all firms rank workers in the same way—also ensures implementation of stable outcomes.

Proposition 4 suggests that stability may be reached even with incomplete information and time frictions when economies are rich. It does not, however, imply uniqueness of these outcomes. In fact, uniqueness in such economies is much harder to guarantee and is more heavily tied with the environments' details, even when market participants are arbitrarily patient. Indeed, even when the economy is rich, suppose one market is far more likely than others, which all occur with equal (small) probability. Assume further that payoffs are such that, in any market, all agents strictly prefer to match with any agent for sure rather than take a fair lottery between their most favored partner and staying unmatched. Consider the following strategy profile. Firms make an offer to their stable match partner in the most likely market. Workers accept their best offer that is superior to their stable match partner in that market, and exit otherwise. Similarly, any firm that is not accepted at $t = 1$ exits immediately. When the probability of the most likely market is sufficiently high, and the economy is large enough, this profile constitutes an equilibrium in weakly undominated strategies.[21] In fact, the profile survives iterated elimination of weakly dominated strategies. It allows for the stable matching to be implemented in the most likely market, but not in others. Thus, the uniqueness apparent under mild refinements from Proposition 2 when information is complete does not carry over to environments with incomplete information.

4.4 Implementing Stable Matchings—Proof Intuition

Before approaching the proof of Proposition 4, we analyze some minimal conditions strategies have to satisfy in a centralized firm-proposing DA mechanism in order to guarantee a stable outcome. This allows us to ignore incentive compatibility hurdles that are due to interim learning. In the proof of Proposition 4, we illustrate how richness ensures that at least some of these strategy profiles, translated into our dynamic game, are incentive compatible for all participants.

In the centralized setting (see Section 2.4), if there is any hope of achieving the complete-information stable matching for any market realization, agents must declare plausible stable match partners acceptable. Furthermore, consider the firms, for example. Since the firm-proposing DA achieves the firm-optimal stable matching for submitted preferences, permuting the ranking of

[21] A firm observing private information inconsistent with the most likely market may be tempted to make an offer to a different worker than prescribed. However, in a large enough market, such an offer would at best generate an approximately fair lottery between the targeted worker and staying unmatched.

agents that are less preferred than a firm's stable match partner would not change the resulting matching. However, it is crucial that agents ranked above *any* plausible stable match partner are, in fact, preferred to that stable match partner. These restrictions suggest a class of strategies.

Formally, for any $l \in \mathcal{F} \cup \mathcal{W}$ of type $\alpha \in \{f, w\}$, let $S_l^\alpha\left(u_l^\alpha(\cdot)\right)$ denote agent l's plausible stable match partners at the outset, using only the information in l's own match utilities.[22] When agent l submits utilities v corresponding to preferences $P(v)$, let $A(v) = \{k : kP(v)\varnothing\} \cup \{\varnothing\}$, where $A(v) \subseteq \mathcal{W} \cup \{\varnothing\}$ if $l \in \mathcal{F}$ and $A(v) \subseteq \mathcal{F} \cup \{\varnothing\}$ if $l \in \mathcal{W}$. That is, $A(v)$ is the union of l's acceptable match partners given $P(v)$ and the option of staying unmatched, namely the "weakly acceptable" partners of l under $P(v)$.

Definition (Reduced DA Strategies)**.** In an economy $\mathcal{E}$, suppose agent $l \in \mathcal{F} \cup \mathcal{W}$ of type $\alpha \in \{f, w\}$ submits utilities v corresponding to preferences $P(v)$. Then v is a reduced DA strategy if:

1. $S_l^\alpha\left(u_l^\alpha(\cdot)\right) \subseteq A(v)$.
2. For each $k, m \in S_l^\alpha\left(u_l^\alpha(\cdot)\right)$, $kP(v)m \Leftrightarrow u_l^\alpha(k) > u_l^\alpha(m)$.
3. For each $k \in A(v) \backslash S_l^\alpha\left(u_l^\alpha(\cdot)\right)$ and each $m \in S_l^\alpha\left(u_l^\alpha(\cdot)\right)$, $kP(v)m \Rightarrow u_l^\alpha(k) > u_l^\alpha(m)$.

The first restriction requires that any plausible stable match partner—using the agent's private information on match utilities—is declared acceptable. The second restriction states that plausible stable match partners are ranked truthfully. The third restriction is perhaps more subtle. To glean intuition, suppose firm k is a "weakly acceptable" match partner for a worker l, but is not a plausible stable match partner. Suppose further that worker l ranks firm k above a plausible stable match partner, firm m, despite preferring m to k. Since m is a plausible stable match partner, in some plausible market, agent l receives an offer from m. Under the declared preferences, such an offer may end up being rejected in favor of an offer from k, which is less preferred and not necessarily part of a stable matching. The third restriction rules out such cases, with similar intuition holding for firms. Importantly, all three restrictions depend only on the *support* of stable matchings, and not on their precise likelihood of occurrence.

We now show the sense by which reduced DA strategies place minimal requirements for securing stability in centralized markets. We consider economies with a finite number of potential markets: strategy profiles that generate the stable matching in each market need to be robust to the underlying distribution over these markets. We say an agent l uses a **rule** if, for any economy $\mathcal{E}$

[22] If $\alpha = f$ (and $l \in \mathcal{F}$), then $u_l^\alpha(k) = u_{lk}^f$; otherwise, $u_l^\alpha(k) = u_{kl}^w$.

containing agent l, the agent uses a strategy that depends only on the set of market participants, the agent's realized match utilities, and the set of potential stable match partners. Agent l uses a **reduced DA rule** if the utilized strategy is a reduced DA strategy.[23]

Lemma 2 (Stable Implementation in Centralized Economies)**.**

1. *If all agents use a reduced DA rule then, for any economy and any realized market, the outcome produced by the DA mechanism is the unique stable matching in that market.*

2. *In any economy, all agents using reduced DA strategies constitutes a Bayesian Nash equilibrium of the game associated with the DA mechanism.*

3. *Suppose an agent $a \in \mathcal{F} \cup \mathcal{W}$ uses a rule that is not a reduced DA rule. Then, there exists an economy for which the DA mechanism produces an outcome that is not stable in some market realization.*

Part 1 of Lemma 2 illustrates the effectiveness of reduced DA rules in generating stable outcomes. Part 2 of the lemma is a strengthening of the second part of Lemma 1. Together with Part 1, it highlights, again, the fact that in centralized matching markets, implementation of stable outcomes can be done through equilibrium, even with incomplete information. Part 3 of the lemma highlights the necessity of the conditions imposed by reduced DA strategies for implementing stable outcomes in any economy.

To prove Proposition 4, we translate reduced DA strategies to dynamic prescriptions. For firms, the translation is straightforward. Whenever firm i submits a reduced DA strategy v, it acts as follows in the decentralized market game. In each period in which firm i is not matched and does not have an offer held by a worker, firm i makes an offer to its most preferred unmatched worker who has not rejected firm i yet according to v (where ties are broken according to the same rules determining $P(v)$ in the centralized setting, depending on the index of the match partner and in favor of matching). When firm i gets rejected by the last acceptable worker (according to v), firm i exits the market.

[23] Reduced DA *rules* impose restrictions on the details of the economy agents can utilize in their strategies. In particular, suppose $\mathcal{E}_1$ and $\mathcal{E}_2$ are two economies with the same set of firms and workers, both containing a market $M = (U, \mathcal{F}, \mathcal{W})$ such that, for some agent $l \in \mathcal{F} \cup \mathcal{W}$ of type $\alpha \in \{f, w\}$, the set of a priori stable matches $S_l^\alpha(u_l^\alpha(\cdot))$ is identical when M is realized in either economy. Then, if agent l uses a reduced DA rule, he or she must use the *same* reduced DA strategy in both $\mathcal{E}_1$ and $\mathcal{E}_2$, whenever observing $u_l^\alpha(\cdot)$.

For workers, there are two aspects of strategies that are important. The first is when to start accepting offers, the second is which offer to accept. The use of weakly undominated strategies implies that when a worker accepts an offer, it must be his favorite available. Thus, each worker has to rank *all* firms in the "right" order. However, in a decentralized market, a worker may accept an offer even if it is not from his most preferred unmatched firm. The translation of a reduced DA strategy v is as follows. At each point in time, a worker accepts the best available offer that he likes at least as much as his most preferred unmatched plausible stable match partner, ranked according to v: the "threshold firm." So, if the reduced DA strategy v only ranks potential stable match partners, the worker accepts an offer as soon as he receives an offer he prefers at least as much as the most preferred unmatched potential stable match partner. In order to mimic the operation of reduced DA strategies within the firm-proposing DA mechanism, we require workers to hold their best available offer as long as the offer is from a firm at least as desirable as the lowest-ranked potential stable match, and reject all other offers.

When moving from a centralized to a decentralized market, agents may also condition their actions on their histories. Specifically, in decentralized markets, given the strategy profile of all other agents, each firm or worker can recalculate and possibly refine their set of *plausible* stable match partners over time. Off path, when an agent observes an event that should occur with probability zero—say, all the agent's stable match partners exit—we assume the agent redefines the set of plausible stable partners to include all remaining agents on the other side of the market.[24]

Decentralized reduced DA strategies are strategies that can be derived as above when allowing agents to submit a new reduced DA strategy every period. This allows agents to take information they accumulate into account. In particular, each firm makes successive offers to workers that are at least as good as her most preferred unmatched plausible stable partner—assessed via each period's posterior—who has not rejected her yet. On the worker side, decentralization has two implications. First, a worker immediately accepts an offer from any firm ranked as high as his most preferred unmatched firm. Second, in analogy to the operation of the DA algorithm, the worker rejects all offers that are not as good as the least preferred plausible stable partner, and holds at most one offer.

In the proof of Proposition 4, we show that beliefs cannot be beneficially manipulated through deviations from decentralized reduced DA strategy profiles when δ is high. The key intuition is

[24]As we show in the appendix, this condition is sufficient for decentralized reduced DA strategies to survive iterated elimination of weakly dominated strategies.

the following. Through deviating, an agent attempts to manipulate beliefs at the possible cost of relinquishing the "true" stable match partner. For example, a firm makes an offer to a lower-ranked worker or a worker rejects its best-held offer. Since all markets have full support, with positive probability, such manipulations can cause an agent to miss out on the best match partner and experience a strict utility loss. Furthermore, the no-cycle property of aligned markets ensures that rejecting a good offer cannot generate a superior offer. In principle, deviations may expedite the time at which an agent is matched. However, for sufficiently high δ, the strictly positive loss from losing the most-preferred match overwhelms the gain due to timing.

4.5 Duration of Decentralized Interactions

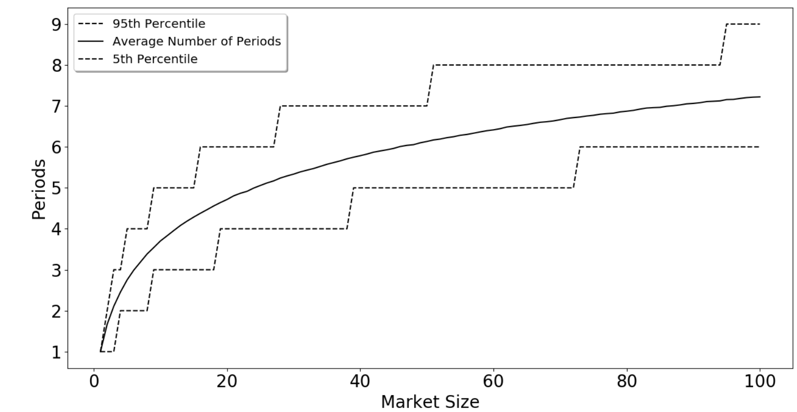
With richness, Proposition 4 guarantees the implementation of stable matchings even in the presence of information and time frictions. One potential concern, however, pertains to the time it takes markets to stabilize. In practice, prolonged durations of instability may entail substantial efficiency losses: the use of unemployment benefits, limited learning-by-doing of the labor force, and so on.

The longest any realized market might take to reach a stable outcome is given by $\min\{F, W\}$. This is due to our alignment assumption. In each period, at least one top-top match is present. Therefore, at least one firm and one worker exit each period until one side of the market is exhausted. Nonetheless, in principle, stable outcomes can be achieved more rapidly for "most" conceivable markets. In order to get a sense of these durations of decentralized interactions, we consider markets in which firms' and workers' preferences are determined uniformly at random from all possible aligned preference profiles. We run 15,000 simulations for each market size—the number of firms and the (equal) number of workers. We assume agents follow decentralized reduced deferred acceptance strategies.[25] Figure 1a illustrates the average number of periods markets take to reach the stable outcome.[26]

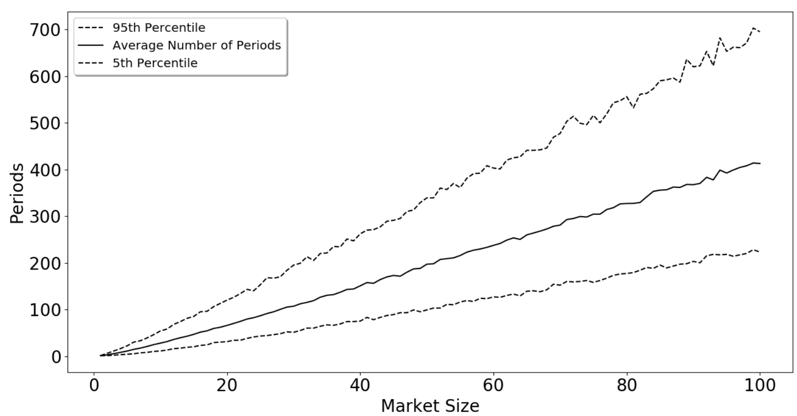
In our simulated markets, the number of periods required to reach stability is far smaller than our upper bound. For instance, for a market with 200 participants (100 firms and 100 workers), the average time to reach stability through equilibrium dynamics is approximately 7 periods.

[25]It is straightforward to see that, in rich economies, all decentralized reduced DA strategy profiles generate the same duration of interactions till the market game ends. In fact, the duration of interactions in each market corresponds to the number of periods the firm-proposing DA algorithm takes in each realized market.

[26]The simulation code can be found at `https://github.com/aaandrew152/DecentralizedMatchingSims`.



(a) Decentralized market game dynamics



(b) Random paths to stability dynamics

FIGURE 1: Duration of activity across market sizes

While there is little work on stabilization dynamics, Roth and Vande Vate (1990) have offered a different approach than ours. They suggest naïve dynamics, whereby at each period, one blocking pair is selected at random and implemented, possibly severing the current pairings of the agents in that blocking pair. The underlying idea is that agents see the current matching, know everyone's preferences—that is, complete information is implicitly assumed—and can therefore infer a viable (myopic) improvement. In our setting, only firms make offers. They do not observe tentative matches, and cannot always infer whether a prospective partner will accept their offer. We therefore consider a modification of Roth and Vande Vate (1990), where only firms make offers to arbitrary workers they prefer to their current match, chosen uniformly at random. Some such offers are then rejected when not directed at blocking partners. In the Roth and Vande Vate (1990) procedure, only one blocking pair is formed each period. This sequencing would mechanically increase the duration of market interactions relative to the one identified in our model, where multiple firms can make simultaneous offers each period. In order to eliminate this mechanical effect, in our simulations of the naïve dynamics, we intoduce another modification by allowing all firms to make offers each period. Specifically, we use 15,000 simulations for each market size for the market game dynamics and 1,000 simulations for each market size for the incomplete information dynamics, allowing multiple firms to make offers to a random partner they prefer in each period. Figure 1b describes the simulation-based durations corresponding to this variant of the Roth and Vande Vate (1990) procedure.

As can be seen, these myopic best-response dynamics lead to substantially longer durations of market interactions. With 200 market participants, the average time to reach stability is approximately 175 periods. In the Online Appendix, we provide analogous results for the original Roth and Vande Vate (1990) process, where one random agent—firm or worker—selects a blocking pair in each period. The resulting durations are substantially longer. In addition, we show that restricting agents to make offers to their most desirable blocking pair as in Ackermann et al. (2011) and Rudov (2022), decreases expected market durations, but still generates market interactions that take far longer than those emerging through equilibrium play.

The simulations suggest three messages. First, strategic stabilization can be quite rapid, even when agents require a lot of information to infer the market-wide stable matching. Second, strategic stabilization is far quicker than naïve stabilization, which entails long durations and leads to participants being mismatched for long periods; see also Rudov (2022). Third, technically, models assuming naïve sequential pairings require modification to allow for incomplete information, balancing strategic and statistical sophistication.

5 Conclusions

We analyze a decentralized market game in which firms make offers that workers react to, allowing for incomplete information and time frictions. We identify when there exists an equilibrium in weakly undominated strategies that yields the complete-information stable matching. This is the case with complete information, when the economy consists of a single market. It is also the case when there are no time frictions. With both uncertainty and time frictions, the economy needs to be sufficiently rich and and market participants have to be sufficiently patient for there to be an equilibrium that yields stability. When both conditions hold, our analysis provides a non-cooperative foundation for the cooperative stability notion.

Taken together, our results indicate that when studying markets, it is crucial to understand characteristics that go beyond the identification of market participants and their preferences. The information available to participants and the plausibility of frictions both play an important role in predicting outcomes. This, in turn, implies that channels by which information can be transmitted among market participants can be a critical element of market design.

6 Appendix – Proofs

The following notation will be useful for several of our proofs. Suppose preferences are aligned with ordinal potential $\Phi = \left(\Phi_{ij}\right)$. Let

$$\begin{aligned}
M^{(1)} &= \{(i,j) \in \mathcal{F} \times \mathcal{W} | (i,j) \in \arg\max_{(i,j)\in\mathcal{F}\times\mathcal{W}} \Phi_{ij}\}, \\
\mathcal{F}_M^{(1)} &= \{j | \exists i \in \mathcal{W} \text{ s.t. } (i,j) \in M^{(1)}\}, \\
\mathcal{W}_M^{(1)} &= \{i | \exists j \in \mathcal{F} \text{ s.t. } (i,j) \in M^{(1)}\}.
\end{aligned}$$

And denote

$$\mathcal{F}^{(2)} = \mathcal{F} \setminus \mathcal{F}_M^{(1)},\ \mathcal{W}^{(2)} = \mathcal{W} \setminus \mathcal{W}_M^{(1)}.$$

The submarket corresponding to $\mathcal{F}^{(2)}$ and $\mathcal{W}^{(2)}$ has aligned preferences and Φ restricted to those firms and workers serves as an ordinal potential. We can replicate the construction above recursively and define, for any k,

$$\begin{aligned}
M^{(k)} &= \{(i,j) \in \mathcal{F}^{(k)} \times \mathcal{W}^{(k)} | (i,j) \in \arg\max_{(i,j)\in\mathcal{F}^{(k)}\times\mathcal{W}^{(k)}} \Phi_{ij}\}, \\
\mathcal{F}_M^{(k)} &= \{j : \exists i \in \mathcal{W} \text{ s.t. } (i,j) \in M^{(k)}\},
\end{aligned}$$

$$\begin{aligned}
\mathcal{W}_M^{(k)} &= \{i : \exists j \in \mathcal{F} \text{ s.t. } (i,j) \in M^{(k)}\}, \\
\mathcal{F}^{(k+1)} &= \mathcal{F} \setminus \mathcal{F}_M^{(k)}, \text{ and } \mathcal{W}^{(k+1)} = \mathcal{W} \setminus \mathcal{W}_M^{(k)}.
\end{aligned}$$

The unique stable matching μ_M can be identified using this recursion: for each k, whenever $(i,j) \in M^{(k)}$, $\mu_M(i) = j$. Any unassigned agents are unmatched under μ_M.

Proof of Proposition 1

1. Suppose preferences are aligned with ordinal potential $\Phi = \left(\Phi_{ij}\right)$. For any submarket $(\tilde{\mathcal{F}}, \tilde{\mathcal{W}}, \tilde{U})$, any pair $(i,j) \in \arg\max_{(i,j)\in\tilde{\mathcal{F}}\times\tilde{\mathcal{W}}} \Phi_{ij}$ constitutes a top-top match, and the top-top match property holds. In addition, suppose μ is a matching different than the unique stable matching μ_M. Reconstruct μ_M by pairing successive top-top matches. Consider the smallest k such that there is a pair (i,j) that is matched under μ_M but not under μ. Then, (i,j) blocks μ and $(i,j) \in M^{(k)}$ form a top-top match in the submarket corresponding to $\mathcal{F}^{(k)}$, $\mathcal{W}^{(k)}$. As this is the first discrepancy between μ_M and μ in the iterative process, the match partners $\mu(i)$ of i and $\mu(j)$ of j are part of the remaining set of firms and workers and hence inferior to $\mu_M(i) = j$ and $\mu_M(j) = i$, respectively. That is, the stable blocking property holds.

2. A rejection cycle is equivalent to the existence of a weak improvement cycle in the two-player game with payoff matrix $((u^w_{ij}, u^f_{ij}))_{i,j}$. Our claim then follows from Voorneveld and Norde (1997).■

Proof of Proposition 2

When $\delta = 1$, the proof follows from Proposition 3. With $\delta < 1$, consider any Nash equilibrium that survives iterated elimination of weakly dominated strategies. At any stage, a worker receiving an offer from the most preferred available firm accepts it immediately. In period 1, any firm $i \in \mathcal{F}^{(1)}_M$ must then make an offer to $\mu_M(i)$, who accepts immediately. Thus, any worker in $\mathcal{W}^{(2)}_M$ accepts an offer from firms in $\mathcal{F}^{(2)}_M$ immediately. It follows that each firm $i \in \mathcal{F}^{(2)}_M$ must make an offer to $\mu_M(i)$ in period 1 as well. Continuing recursively, we get that all firms matched under μ_M must make offers to their partners under μ_M in period 1, and those get accepted immediately. Firms or workers that are not matched under μ_M must then exit the market in period 1 when $\delta < 1$. In particular, any profile surviving iterated elimination of weakly dominated strategies generates the matches prescribed by μ_M in period 1. ■

Proof of Proposition 3

Since preferences are aligned and everyone is acceptable to everyone, in every period with unmatched agents, there is either a top-top match that is formed, or only agents on one side of the market are unmatched. Thus, DA strategies generate a market matching in finite time. From the convergence of the Gale-Shapley algorithm to a stable matching, it follows that DA strategies yield the stable matching. We now show that DA strategies constitute a Bayesian Nash equilibrium.

Workers can deviate in two ways. First, a worker j can reject an offer from firm i instead of holding it. From the no-cycle property, such a rejection cannot launch a chain generating a superior offer for j. In addition, if i is a plausible stable match partner, such a rejection may lead j to forgo his best offer in some market. Such a deviation could therefore be profitable only if it makes the worker sufficiently better off in some market realization in which firm i is not his stable match partner. However, in any market, it cannot be that i is strictly better than j's stable match partner, as then j should never receive an offer from i (indeed, by the construction of DA strategies, firm i and worker j would form a blocking pair to the stable matching). The second potential deviation of a worker is the acceptance of an offer that is not from his most preferred unmatched firm. However, workers are made better off over time, as they receive new offers. Therefore, accepting an offer early cannot be a profitable deviation when $\delta = 1$.

Consider the firms. Suppose firm i deviates and makes an offer to worker j who is not the most preferred among workers who have not rejected i yet. Since $\delta = 1$, if there is a market in which i strictly benefits from this deviation, it must be that i ends up matching with a strictly preferable worker. Suppose the resulting market matching (assuming all other agents follow DA strategies) is μ. The matching μ has the property that the set of firms F', who prefer this match to the stable match μ_M, is non-empty, as it contains at least firm i. By the Blocking Lemma, there exists a blocking pair (i', j') with i' not in F' such that j' is matched in μ to a firm in F' Roth and Sotomayor (1992). Since i' and j' follow DA strategies, i' must have made an offer to j', a contradiction.

We now show uniqueness of Bayesian Nash equilibrium matching outcomes under iterated elimination of weakly dominated strategies. Whenever there is at least one remaining firm, a worker's strategy to exit in period t is weakly dominated by a strategy under which the worker stays in period t and exits in period t+1 upon not receiving an offer. In particular, in any equilibrium surviving iterated elimination of weakly dominated strategies, workers cannot exit in any market unless there are no remaining firms. Similarly, if a worker receives an offer from her top remaining firm, she cannot reject that offer or exit: a strategy incorporating either action would be weakly dominated by a strategy that dictates accepting that offer instead. This implies that if firm i proposes to a worker who considers i to be his top remaining firm, i must eventually be accepted. Last, a worker cannot accept an offer from a firm that is not from their top remaining firm: accepting such an offer would be weakly dominated by holding the offer for one period, and allowing for a better offer in the next period.

Our discussion suggests that in any equilibrium surviving iterated elimination of weakly dominated strategies, workers never exit prior to the market concluding, cannot accept non-top offers, and must eventually accept a top offer. It follows that workers either remain in the market indefinitely, or accept their top remaining offers. Since realized matches are a result of workers accepting their top remaining offers, alignment implies that those matches must coincide with the unique stable matching. Furthermore, in any equilibrium, workers cannot remain unmatched forever: they would be better off exiting at some point. Therefore, firms must eventually make offers that are accepted on path, which ultimately culminates in the stable matching. ■

Proof of Lemma 2

1. Assume all agents use a reduced DA rule and suppose $\mathcal{E}$ is an economy with a market realization in which the outcome μ' is different than the stable matching μ_M. By Proposition 1, there exists a pair (i', j') that blocks μ with $\mu_M(i') = j'$. This implies that $u^f_{i'j'} > u^f_{i'\mu(i')}$ and $j' \in \mathcal{S}^f_{i'}(u^f_{i'\cdot})$. Hence, if firm i' uses a reduced DA strategy, she must rank j' above $\mu(j')$. For worker j', since $i' \in \mathcal{S}^w_{j'}\left(u^w_{\cdot j'}\right)$, j' cannot be matched with someone other than i' unless, under DA, he receives a better offer. However, $u^w_{i'j'} > u^w_{\mu(j')j'}$ implies that j' does not reject i''s offer, in contradiction.

2. Assume all agents follow a profile v of reduced DA strategies. For firms, since truthful revelation is a weakly dominant strategy in the firm-proposing DA algorithm, which the centralized market emulates, no deviation can be strictly profitable.

Suppose a worker j has a strictly profitable deviation to a strategy σ_j. If σ_j is also a reduced DA strategy, then by Part 1 above, the outcome is unchanged, in contradiction. Suppose, then, that σ_j yields a matching μ such that $u^w_{\mu(j)j} > u^w_{\mu_M(j)j}$. Then, by Proposition 1, there exists a pair (i', j') that blocks μ such that $\mu_M(i') = j'$. First, it is clear that $j' \neq j$ since j strictly prefers μ to μ_M. Since both i' and j' submit reduced DA strategies, i' must rank j' above $\mu(i')$. Hence, it must be that j' rejects i' through the centralized mechanism, contradicting the fact that (i', j') is a blocking pair under μ.

3. Suppose $a \in \mathcal{F} \cup \mathcal{W}$ is an agent who does not use a DA rule. That is, agent a ranks some agent (including potentially the possibility of staying unmatched, $\varnothing$) as preferred to a plausible stable match partner when match utilities prescribe otherwise. Whenever there is only one worker or only one firm, the claim follows trivially. Assume then that $|\mathcal{F}|, |\mathcal{W}| \geq 2$.

Certainly, if a ranks a plausible stable match partner as unacceptable, then the centralized outcome in some markets would be unstable.

Suppose that $a \in \mathcal{W}$ ranks a plausible stable match partner i below a firm i', who is not a plausible stable match partner when observing $u^w_{\cdot a}$, and the set of plausible stable matches is S. Assume $u^w_{ia} > u^w_{i'a}$. Let $j \in \mathcal{W}$ be another worker (other than a).

Consider an economy in which there are three markets characterized by match utilities $U, \tilde{U}$, and $\hat{U}$ in which, in the corresponding stable matchings, all agents A other than i, i', a, and j are prescribed to be matched to agents in A or remain unmatched. It therefore suffices to focus on match utilities corresponding to agents $\{i, i', a, j\}$.

We construct U and $\tilde{U}$ so that they satisfy the following:[27]

a. Firm i' cannot distinguish between the two markets, while all other agents can.

b. Firm i' prefers worker a to worker j in both markets.

c. Under U, i and a, and i' and j, are part of the stable matching, while under $\tilde{U}$, i' and a are part of the stable matching.

d. Under $\tilde{U}$, i' is both a's and j's most preferred firm.

$\hat{U}$ is such that $\hat{u}^w_{\cdot j} = \tilde{u}^w_{\cdot j}$, so that worker j cannot distinguish $\tilde{U}$ from $\hat{U}$, and j is the most preferred worker for i'. Each of the remaining markets in the economy is one in which a's match utilities are given by $u^w_{\cdot a}$ and the stable matching is an element $i'' \in S \setminus \{i\}$.

If the stable matching is achieved under $\hat{U}$, worker j must rank firm i' as acceptable when observing $\hat{u}^w_{\cdot j} = \tilde{u}^w_{\cdot j}$. Therefore, if the stable matching is achieved under $\tilde{U}$, it must be the case that i' ranks a higher than j (and acceptable) when observing $u^f_{i'\cdot}$. But then, under U, it cannot be the case that the stable matching is established. Indeed, the centralized mechanism generates a stable matching for the submitted preference rankings, and i' and a would form a blocking pair.

A similar construction follows if $a \in \mathcal{W}$ ranks a plausible stable match partner i below a less preferred plausible stable match partner i' when observing $u^w_{\cdot a}$ and the set of plausible stable matches is S. Furthermore, analogous constructions follow for firms. ■

Before providing the proof of Proposition 4, we introduce a broader class of economies satisfying what we term *generalized richness*.

We use $\mu_M(U)$ to denote the unique stable matching in a realized market with match utilities U. Let $M_t \subseteq (\mathcal{F} \cup \varnothing) \times (\mathcal{W} \cup \varnothing)$ denote the matches formed at time t, including firms and workers who

[27]These conditions are consistent with alignment. Indeed, assuming, without restriction, that $u^w_{kl}, \tilde{u}^f_{kl} > 2$ for any $k \in \{\varnothing, a, j\}$ and $l \in \{\varnothing, i, i'\}$, the reader can think of the following manifestation of $U, \tilde{U}$ in which we summarize preferences through the following two matrixes, where the first number in each entry corresponds to the firm's preference and the second number to the appropriate worker:

U:

	a	j
i	u^w_{ia}, u^w_{ia}	$u^w_{ia} - 1, u^w_{ia} - 1$
i'	$u^w_{i'a}, u^w_{i'a}$	$u^w_{i'a} - 1, u^w_{i'a} - 1$

$\tilde{U}$:

	a	j
i	$\tilde{u}^f_{i\varnothing} - 1, \tilde{u}^w_{\varnothing a} - 1$	$\tilde{u}^f_{i\varnothing} - 2, \tilde{u}^w_{\varnothing j} - 1$
i'	$u^w_{i'a}, \tilde{u}^w_{\varnothing a} + 1$	$u^w_{i'a} - 1, \tilde{u}^w_{\varnothing j} + 1$

.

exit unmatched, and let the set of agents who exited the market up to, but excluding, period t be

$$\mathcal{X}^t \equiv \{j \mid \exists k \text{ s.t. } (j,k) \in M_\tau \text{ for some } \tau < t\} \cup \{i \mid \exists l \text{ s.t. } (l,i) \in M_\tau \text{ for some } \tau < t\}.$$

Last, let $M_t^F \subseteq \mathcal{F} \times \varnothing$ be the set of firms who leave the market in the first stage of period t.

At the start of period t, each active firm i observes a history consisting of her (timed) offers, workers' responses to those offers, denoted by r for rejection and h for holding (we denote an offer to no worker as an offer to $\varnothing$ that is immediately rejected), and the (timed) exits of agents:[28]

$$h_{t,i}^f \in ((\mathcal{W} \cup \varnothing) \times \{r,h\})^{t-1} \times \prod_{\tau=0}^{t-1} M_\tau.$$

Each unmatched worker acts in the interim stage of every period t and observes a history that consists of all (timed) offers he received, including those at time t, a (timed) sequence of offers he has held, and the (timed) set of agents that have left the market up to and including time t:

$$h_{t,j}^w \in \left(2^{\mathcal{F}}\right)^t \times \left(2^{\mathcal{F}}\right)^t \times \prod_{\tau=0}^{t-1} M_\tau \times M_t^F.$$

For a given prior distribution G over utility realizations, for any private information $u_l^\alpha(\cdot)$ of agent l with $\alpha \in \{f,w\}$ regarding the realized market, let $G(u_l^\alpha(\cdot))$ denote the posterior distribution over utility realizations. Let $\mathcal{S}_l^\alpha(u_l^\alpha(\cdot)) = \left\{\mu_M(U)(l) \mid U \in \text{supp } G(u_l^\alpha(\cdot))\right\}$ denote the set of all ex-ante plausible stable match partners of agent l. That is, agents that could conceivably be part of a stable matching, under the distribution over market match utilities updated by the private information $u_l^\alpha(\cdot)$. Analogously, given the strategies played by all agents, $\mathcal{S}_l^\alpha(u_l^\alpha(\cdot), h_{t,l}^\alpha)$ denotes the set of all interim potential stable match partners given agent l's available information at t.

A reduced DA strategy for a firm i is *minimal* if the following condition is satisfied:

$$A(v) = S_i^f(u_i^f(\cdot))$$

Intuitively, firm i must "skip" any workers that cannot be stable match partners. Similarly, a reduced DA strategy for a worker j is *minimal* if the following condition is satisfied:

$$\text{If } \forall k \in S_j^w(u_j^w(\cdot)),\ u_j^w(k) > u_j^w(i) \implies i \notin A(v)$$

Namely, any firm i ranked by j below all of j's stable match partners is listed as unacceptable. We call a decentralized reduced DA strategy a **decentralized minimal DA strategy**, if in each stage, the reduced DA strategy is minimal.

[28] An offer of firm i to worker j that is held from period t to t' is recorded as an offer made in periods $t, t+1, ..., t'$ that is held by the worker in each of these periods.

Assumption 1 *Suppose all agents use decentralized minimal DA strategies. Consider any firm i and market realization with match utilities U. At each period t, for all available workers j ranked below firm i's most-preferred potential stable match partner that has not rejected the firm yet, either*

1. *there exists* $\tilde{U} \in supp\ G\left(u^f_{i\cdot}, h^f_{t,i}\right)$ *such that* $\tilde{u}^w_{ij} > \tilde{u}^w_{\mu_M(\tilde{U})(j)j}$, *or*
2. *for all* $\tilde{U} \in supp\ G\left(u^f_{i\cdot}, h^f_{t,i}\right)$, $\tilde{u}^w_{ij} < \min\{\tilde{u}^w_{kj} : k \in \mathcal{S}^w_j\left(\tilde{u}^w_{\cdot j}, h^w_{t,j}\right)\}$.

Assumption 1 ensures that when all market participants follow decentralized minimal DA strategies, a firm has no incentive to make an offer to a worker who is ranked below her favorite unmatched plausible stable partner that has not rejected her yet. If the firm makes such an offer, Assumption 1 guarantees the firm runs the risk of one of two eventualities. The first is that the firm might have her offer held or accepted, as it is better than the stable match of the worker in the realized market. The second occurs if the worker uses a decentralized minimal DA strategy that specifies firms that are less preferred than all plausible stable matches as unacceptable. In that case, the firm will be rejected immediately. Making an offer then has no benefit.

Denote by $\tilde{W}^t_i$ the set of workers who have not rejected firm i before period t, by $\tilde{F}^t_j$ the set of firms that have not made an offer to worker j up to (and including) period t and that he weakly prefers to any firm that has made him an offer till then, and by $\mathcal{X}^t$ the set of agents who have exited by period t. Let

$$\begin{aligned} \mathcal{BS}^f_i\left(u^f_{i\cdot}, h^f_{t,i}\right) &\equiv \left\{j | j \in \mathcal{S}^f_i\left(u^f_{i\cdot}\right) \cap \tilde{W}^t_i\right\} \setminus \mathcal{X}^t, \\ \mathcal{BS}^w_j\left(u^w_{\cdot j}, h^w_{t,j}\right) &\equiv \left\{i | i \in \mathcal{S}^w_j\left(u^w_{\cdot j}\right) \cap \tilde{F}^t_j\right\} \setminus \left(\mathcal{X}^t \cup M^F_t\right). \end{aligned}$$

Assumption 2 *Suppose all agents follow decentralized minimal DA strategies.*[29] *Let U be in the support of G and l be an agent of type* $\alpha \in \{f, w\}$. *Assume that in period t, if* $\alpha = w$, *worker l has not received offers from all but one of the remaining firms. For each* $j, k \in \mathcal{BS}^\alpha_l\left(u^\alpha_l(.), h^\alpha_{t,l}\right)$, *if* $u^\alpha_l(j) > u^\alpha_l(k)$ *and* $k \in S^\alpha_l\left(u^\alpha_l(.), h^\alpha_{t,l}\right)$, *then* $j \in S^\alpha_l\left(u^\alpha_l(.), h^\alpha_{t,l}\right)$.

Assumption 2 poses that when all market participants follow decentralized minimal DA strategies, the ordering of offers and matches does not convey information in and of itself. Put differently, it ensures firms cannot cross out their favorite available plausible stable match partner from the set of perceived plausible stable matches $\mathcal{S}^f_i(u^f_{i\cdot}, h^f_{t,i})$ when updating based on worker

[29] Any decentralized minimal DA strategy profile leads to the same learning pattern pertaining to stable matches, and so the assumption's requirement is not affected by which particular profile is used.

matches, exits, and rejections (generating $\mathcal{BS}_i^f(u_{i\cdot}^f, h_{t,i}^f)$). An analogous implication holds for workers. An exception occurs when a worker receives $\tilde{F}-1$ offers, where $\tilde{F}$ is the number of firms remaining. Suppose the worker does not receive an offer from his most-preferred firm. Then, since a top-top match is made in each period, it must be that the remaining firm has formed a top-top match and will exit. Thus, the worker believes the most-preferred firm of the $\tilde{F}-1$ that had made him an offer is his stable partner.

An economy satisfies **generalized richness** if it satisfies Assumptions 1 and 2. As in our definition of rich economies in the body of the paper, generalized richness refers to the *support* of potential match utilities. It does not rule out probabilistic updating on the likelihood of different agents being one's stable match in the realized market. While generalized richness is still somewhat restrictive, it allows for a rich class of settings. As we show in Proposition 4* below, generalized richness is, indeed, a generalization of the richness assumption in the main body of the paper. Generalized richness also allows for assortative economies, where the distribution of markets is supported on all preference profiles in which, say, firms (similarly workers) have identical ordinal rankings of workers (similarly, firms).

Proposition 4* (Generalized Richness–Implementation)**.** *Suppose the economy $\mathcal{E}$ satisfies generalized richness. For sufficiently high δ, there is a Bayesian Nash equilibrium in weakly undominated strategies of the decentralized market game that implements the unique stable matching in each supported market.*

Proof of Proposition 4*

We first show that, for sufficiently high δ, a decentralized minimal DA strategy is a worker's best response when: 1. all other workers use decentralized minimal DA strategies, specifying only plausible stable partners as acceptable; and 2. firms use decentralized minimal DA strategies.

Indeed, at each period t, for sufficiently high δ, a worker cannot benefit by exiting the market whenever a plausible stable partner is still available, nor from accepting an offer from a firm who is not his most preferred plausible stable match. A worker with offers at hand cannot benefit by holding on to firms other than his most preferred. This follows from the no-cycle property. Indeed, rejection of firms cannot generate the arrival of an offer from a preferred firm, and decentralized minimal DA strategies ensure that rejected firms will not make future (repeat) offers.

We proceed by showing that there exists a unique decentralized minimal DA strategy for any firm. To begin, under DA strategies, for any firm and any preference ranking the set of plausible stable match partners is fixed. The second restriction in the definition of DA strategies requires all plausible stable match partners to be ranked truthfully. Furthermore, minimality requires implausible stable match partners to be omitted from the firm's list of acceptable match partners. Then, only plausible stable match partners can be listed, and they must be ranked truthfully.

We now show that, for sufficiently high δ, the unique minimal DA strategy is a firm's best response whenever: 1. workers use decentralized minimal DA strategies; and 2. firms use their unique minimal DA strategies.

Consider first a firm i that in period t has no outstanding offers, and whose updated strategies suggest worker j as the most preferred stable match. There are two kinds of deviations from a decentralized minimal DA prescription: make no offer, or make an offer to some other worker k who is ranked below j.

If firm i does not make an offer at period t, there are three potential implications. First, if making an offer according to any decentralized minimal DA strategy would not have affected market participants' history following period t,[30] then the only effect of this deviation could be the prolonging of its match creation. If not making an offer affects certain participants' histories, then due to Assumption 2, this cannot affect the firm's final match. Again, such a deviation can only prolong the timing of that final match. Finally, suppose that the firm's most preferred plausible stable partner receives offer from all other firms remaining on the market at period t. Then, the worker will accept the best offer of those immediately, even if he prefers firm i (see our discussion following Assumption 2), in which case firm i is made strictly worse off by the deviation.

Suppose firm i makes an offer to a worker k who is ranked lower than her most preferred plausible stable match j. By Assumption 1, the firm could be immediately rejected, in which case she does not benefit. Alternatively, with positive probability, her offer will be held or accepted by k in a market in which she would have otherwise gotten a preferable worker. Such a deviation can never lead to a preferable ultimate match from the incentive compatibility inherent in the firm-proposing DA algorithm. Indeed, such a deviation would be tantamount to submitting an untruthful preference list when the firm-proposing DA algorithm is used (as, from Assumption 2, such an offer will not make other participants change their effective rank orderings). However, revealing preferences truthfully is a dominant strategy for firms. ■

Proof of Proposition 4

We show that rich economies satisfy generalized richness. By Proposition 4*, the result follows.

[30]For instance, in the case in which any such strategy suggests an offer to j, who gets matched to another firm in that period with probability 1.

Alignment implies that agents' match utilities are a monotonic transformation of the elements of their respective row or column of the ordinal potential matrix. Without loss of generality, we assume match utilities take values in $\{\varepsilon, 1, 2, \ldots, n\}$, where $n = W$ or $n = F$ depending on whether the agent is a firm or a worker. $\varepsilon \in (0,1)$ corresponds to the utility from remaining unmatched. An equivalent construction follows for any other set of distinct values that contains at least as many elements. Our construction also follows in much the same way if utility values differ across market participants. For any ordinal potential Φ, the mapping from entries Φ_{ij} to utility values for a firm i is as follows. Let $j_1 = argmax_j \ \Phi_{ij}$, $j_2 = argmax_{j \neq j_1} \ \Phi_{ij}$, etc. Then, $u^f_{ij_1} = n, u^f_{ij_2} = n - 1$, and so on. Worker utility values are mapped similarly.

In what follows, we construct ordinal potentials that rationalize each history an agent observes and guarantee generalized richness. Our arguments are constructive: we identify a set of ordinal potentials consistent with preferences and histories any agent might observe when called upon to act. However, the precise values we place on entries of these ordinal potentials are of no importance—they simply allow us to identify the profile of plausible preference rankings that can then be translated directly to match utilities. Notably, our construction is designed for on-path play. It requires all agents to have followed reduced DA strategies.

First, we consider an arbitrary firm. We begin by establishing restrictions on the ordinal potential Φ that correspond to the public history, namely the sequence of market exits. It is useful to note that, due to the top-top match property, at least one firm and worker pair exit each period until only firms or only workers remain.

For any firm i and worker j that match and exit in period 1, set $\Phi_{ij} = 0$. Next, for any firm i and worker j that match in period 2, choose an arbitrary firm i', who exited in period 1. Set $\Phi_{i'j} = -j$ and $\Phi_{ij} = -(n+1)$. This restriction rationalizes behavior in these two periods. Indeed, with such an ordinal potential, reduced DA instructs firm i to make an offer to worker j in period 1. Worker j holds the offer but does not accept, in the hope that firm i' will make him an offer. Upon observing firm i''s exit in period 1, worker j accepts his second-best firm i. This specification leads to an additional constraint: the remaining elements in the row corresponding to firm i and the column corresponding to worker j must be lower than $-(n+1)$. We continue this construction iteratively through time. In period k, for any firm i and worker j pair that exits, we set $\Phi_{ij} = -(k-1)(n+1)$ and $\Phi_{i'j} = -(k-2)(n+1) - j$, where firm i' is an arbitrary firm that exited in the previous period. This specification rationalizes the public information available in the economy at any period.

Consider now firm i at some period t, with history $h^f_{t,i}$. The firm knows her own preference ranking, as well as whether workers she has made offers to held or rejected her offers over time. We impose further restrictions on the ordinal potential to accommodate this additional information and rationalize $h^f_{t,i}$. First, the firm's preference ranking implies that the ordering of Φ_{ix} over x is determined by firm i's preferences.

In the second case, we only consider situations where every worker that firm i has made an offer to has rejected the firm. This is without loss of generality, as otherwise the firm has no strategic choice and the requirements for generalized richness are trivially satisfied.

If worker j held firm i's offer for some number of periods, then exited with a firm i' in period $t' \leq t$, $\Phi_{i'j}$ is specified by the above construction. Namely, $\Phi_{i'j} = -(t'-1)(n+1)$. This implies that firm i would have been chosen by worker j had firm i' been unavailable. Furthermore, it ensures any remaining workers rank below worker j in firm i's preferences. This specification determines $\Phi_{i'j'}$ for any worker j' that exited before t' so that firm i' does not make an early offer to worker j. Importantly, the number of rounds that the firm's offer is held is rationalized by the above construction: worker j accepts the offer from firm i' only when firm i' is worker j's most preferred remaining firm.

Similarly, a worker j may hold firm i's offer for some number of periods, then reject the firm's offer without exiting at period $t' \leq t$. With reduced DA strategies, this implies worker j initially preferred firm i's offer to all other received offers. However, in period t', some firm i' made the worker an offer preferable to firm i's offer. Worker j remains in the market only if there exists another firm i'' he prefers to firm i' that is a plausible stable match. One path consistent with firm i's observed history has firm i'' exiting in period $t-1$. From our specification thus far, $\Phi_{i''j'}$ is determined, where worker j' is the worker with whom firm i'' exited. Firm i can also consistently expect worker j to exit with firm i' in period t. We thus let $\Phi_{i'j} = -(t-1)(n+1)$ and $\Phi_{ij} = -(t-1)(n+1) - j$. Again, the fact that worker j remains in the market for $t - t'$ periods is accounted for by our specifications up to now.

The restrictions imposed until now are consistent with firm i's sequence of offers through period t. We add a restriction: for all workers j to whom firm i has not made an offer, $\Phi_{ij} \leq -t(n+1)$. This constraint is consistent with the requirements placed so far.

For other entries of the potential, we require that remaining workers have not received offers from their most-preferred remaining firms. Additionally, if there is more than one remaining firm, we set the potential so that no remaining workers received offers from all but one firm. Specifically, for each worker j who has not exited the market by period t, take a firm i' who exited in period

$t-1$ and set $\Phi_{i'j} = -(t-2)(n+1) - j$. Then, each unmatched worker waited for an offer from firm i' in period $t-1$.

Since the economy is rich, at least one market occurring with positive probability has an ordinal potential that satisfies all the restrictions imposed above.

We now show that the restricted set of ordinal potentials that satisfy the above restrictions are sufficient for generalized richness.

For Assumption 1, take any worker j who is not preferred to firm i's most preferred worker and who has not rejected her yet at period t. Consider ordinal potentials under which worker j might accept an offer made by firm i. Without loss of generality, let firm i prefer worker 1 to worker 2 and so on. Then, indeed, we can add the restrictions $\Phi_{ij} = -n(n+1) - j$ and $\Phi_{i'j} = -n(n+1) - j - i'$ for each firm i' still in the market. With this specification, every remaining worker j immediately accepts offers made by firm i, thereby satisfying assumption 1. Richness implies that potentials satisfying these additional restrictions are in the support of firm i's beliefs.

As for Assumption 2, consider an ordinal potential where firm i's most preferred remaining worker j accepts the firm's offer. Set $\Phi_{ij} = -(t-1)(n+1)$. By design, every firm-worker pair with $\Phi_{ij} > -(t-1)(n+1)$ contains at least one agent that has exited in a previous period. Therefore, firm i and worker j constitute a top-top match in period t. In particular, worker j will immediately accept firm i's offer. Again, richness implies that potentials satisfying these additional restrictions are in the support of firm i's beliefs.

Consider now a worker j at any arbitrary period t. The worker knows his own preference ranking and the sequence of firms that have made him offers up to time t, captured by the history $h^w_{t,j}$. As for firms, we find a subset of ordinal potentials that rationalize the worker's preferences and observed history. We will use the following claim.

Claim *Suppose all agents follow reduced DA strategies. Assume that, in period t, worker j rejects an offer from firm i. Then, if in period $t' > t$ firm i exits and worker j remains, there exists a firm i' that has not made an offer to worker j that also exits in period t'.*

Proof of Claim: Assume Φ is the governing potential in the realized market. Under reduced DA strategies, if a worker j rejects firm i, he must have received an offer from a preferred firm. Suppose that, in period t', the worker holds an offer from firm i^*. Let $\tilde{F}$ and $\tilde{W}$ be the sets of firms and workers that have not exited before period t', respectively. Any pair in $\arg\max_{(\tilde{\imath},\tilde{\jmath})\in\tilde{F}\times\tilde{W}} \Phi_{\tilde{\imath}\tilde{\jmath}}$ constitutes a top-top match that exits in period t'. In particular, there is a firm i' that exists with a worker j'. Suppose firm i' made an offer to worker j at some period. From the definition of DA strategies, it follows that $\Phi_{i^*j} > \Phi_{i'j}$ and $\Phi_{i'j} > \Phi_{i'j'}$, in contradiction to firm i' and worker j' being a top-top match in period t'.

We restrict ordinal potentials to rationalize exited firm-worker pairs. Namely, we construct an "optimistic" ordinal potential consistent with the public history, where any worker believes his top firm remaining in the market is still achievable. Since all ordinal potentials occur with positive probability, the worker prefers to follow a decentralized reduced DA strategy to avoid risking losing his top firm. Specifically, suppose firm i and worker j' exit in period t. We consider two cases. If firm i had made an offer to worker j before, we set $\Phi_{ij'} = -(n+1)^2$. If firm i had not made an offer to worker j before, we set $\Phi_{ij'} = -(t-1)(n+1)$.

In each period, either no firm makes an offer to worker j, or at least one firm does. Since the worker has not exited prior to period t, he could not have received an offer from his most preferred remaining firm in any prior period.

Suppose available firms $i_1, i_2, \dots, i_k$ have not made offers to worker j in any period $t' \leq t$ for $t > 1$. Then, worker j infers that each of these firms preferred other available workers in previous periods. Set their preference rankings as follows. In period 1, take a matched firm-worker pair, with worker j_1. For every firm $i \in \{i_1, i_2, \dots, i_k\}$ set $\Phi_{ij_1} = -i$. That is, worker j assumes that every such firm made an offer to the exiting worker. In period 2, there must be a worker j_2 who exits with a firm that never made worker j an offer. Indeed, if worker j rejected an offer in period 1, such a worker j_2 is guaranteed by the claim above. If worker j did not receive any offers in period 1 or is holding his period 1 offer, such a worker j_2 is guaranteed from the top-top match property of aligned markets. Then, for every firm $i \in \{i_1, i_2, \dots, i_k\}$, set $\Phi_{ij_2} = -(n+1) - i$. Repeat this construction up to period t: in each period $l < t$ take a worker j^l who exited in period l, and set $\Phi_{ij^l} = -(l-1)(n+1) - i$ for all firms $i \in i_1, i_2, \dots, i_k$. This construction also generates the additional restriction that all remaining potential entries involving two agents who have not yet exited be below $-(t-1)(n+1)$. Firms previously rejected by worker j are placed at the "bottom" of the ordinal potential in accordance with their match utilities to ensure that the resulting ordinal potential is consistent with worker j's preferences as described in the previous section.

Suppose worker j has offers from firms $\tilde{\imath}_1, ..., \tilde{\imath}_m$ in period t, which may include any offer held from period t. Without loss of generality, suppose worker j prefers firm $\tilde{\imath}_1$ to firm $\tilde{\imath}_2$ to firm $\tilde{\imath}_3$ and so on. We consider two cases. First, suppose firm $\tilde{\imath}_1$ is worker j's most preferred remaining firm. In that case, reduced DA directs worker j to accept the offer. Since generalized richness does not impose any restrictions in this case, any specification of a consistent potential would do. Second, suppose there exists an available firm i that worker j prefers to firm $\tilde{\imath}_1$. Decentralized reduced DA directs worker j to hold $\tilde{\imath}_1$'s offer and reject the rest. For each rejected firm $i_v = \tilde{\imath}_2, \ldots, \tilde{\imath}_k$, set $\Phi_{i_v j} = -(n+1)^2 + u^w_{i_v j}$. As for firm $\tilde{\imath}_1$ and the most preferred firm i, set $\Phi_{\tilde{\imath}_1 j} = -t(n+1)$ and $\Phi_{ij} = -(t-1)(n+1) - i - 1$.

Next, we show that both of the generalized richness assumptions hold for the workers. Assumption 1 holds immediately, as it only places restrictions on firms' beliefs.

Consider Assumption 2. If there exists only one remaining firm, assumption 2 is satisfied automatically. Otherwise, if worker j has received an offer from his most preferred firm, there is no remaining further preferred firm so assumption 2 holds, as suggested above. Last, suppose worker j has not received offers from all but one of the remaining firms and has not received an offer from his most preferred firm. Then, there exist two distinct firms, say i and i', with i preferable to i', which have not made worker j an offer yet. Consistent with the construction above, we can set $\Phi_{ij} = -(t-1)(n+1) - i$ and $\Phi_{i'j} = -(n+1)^2 + u^w_{i'j}$.

The above construction is consistent with the worker's observed history, which implies that i' exits this period, and i makes an offer to worker j next period. Therefore, Assumption 2 is satisfied.

Since Assumptions 1 and 2 are satisfied, richness implies generalized richness. Proposition 4* then implies that an equilibrium implementing the stable outcome market by market exists. ■

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